Solutions

1. (a) For any $a \neq 0$, it is clear that f is differentiable at a. And for $a = 0$,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} |x| = 0 \implies f'(0) = 0.$$

Observe that we actually have $f'(x) = 2|x|$ for every $x \in \mathbb{R}$.

(b) For $x > 0$, we have $f(x) = \log x$. So, for any $a > 0$, $f'(a) = 1/a$. For $x < 0$, we have $f(x) = \log(-x)$. So, for any $a < 0$, $f'(a) = 1/(-a) \cdot (-1) = 1/a$. Thus, $f'(a) = 1/a$ for every $a \neq 0$.

(c) $f(x) = \lfloor x \rfloor \sin^2(\pi x)$, $x \in \mathbb{R}$. For any $a \in (n, n+1)$ (where $n \in \mathbb{Z}$) it is clear that $f'(a)$ exists and we can easily calculate $f'(a)$. What remains to find is whether f is differentiable at the integers. For any $n \in \mathbb{Z}$, observe that

$$\lim_{x\to n+} \frac{f(x)-f(n)}{x-n} = \lim_{x\to n+} \frac{n\sin^2(\pi x)}{x-n} = \lim_{(x-n)\to 0+} \frac{n\sin^2(\pi(x-n))}{x-n} = 0$$

and similarly

$$\lim_{x\to n-} \frac{f(x)-f(n)}{x-n} = \lim_{x\to n+} \frac{(n-1)\sin^2(\pi x)}{x-n} = 0.$$

So, $f'(n) = 0$ for every $n \in \mathbb{Z}$. We actually have $f'(x) = \pi \lfloor x \rfloor \sin(2\pi x)$ for every $x \in \mathbb{R}$.

(d) Using part (c), $f'(x) = (x\sin^2 \pi x)' - \pi\lfloor x \rfloor \sin(2\pi x) = \sin^2 \pi x + \pi(x - \lfloor x \rfloor)\sin(2\pi x)$.

2. For any $a \neq 0$, it is clear that both f and g are differentiable at a. To study their differentiability at 0, note that

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \sin\frac{1}{x}, \text{ which does not exist,}$$

and,

$$\lim_{x\to 0} \frac{g(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0, \text{ by Sandwich theorem.}$$

3. Clearly[1], $f'(x) = 1/(1+x^2)$ if $|x| < 1$, and $1/2$ if $|x| > 1$. We shall now check whether the derivative exists at $x = \pm 1$. We have

$$f'_+(1) = \lim_{x\to 1^+} \frac{\frac{\pi}{4} + \frac{x-1}{2} - \frac{\pi}{4}}{x-1} = \frac{1}{2},$$

$$f'_-(1) = \lim_{x\to 1^-} \frac{\tan^{-1} x - \frac{\pi}{4}}{x-1} = \frac{d}{dx}(\tan^{-1} x)\Big|_{x=1} = \frac{1}{2}.$$

[1] To find the derivative of $\tan^{-1} x$, you can start from the definition. Substitute $y = \tan^{-1} x$, and $b = \tan^{-1} a$.

So, $f'(1) = 1/2$. We also have

$$f'_+(-1) = \lim_{x \to -1^+} \frac{\tan^{-1} x + \frac{\pi}{4}}{x+1} = \frac{d}{dx}(\tan^{-1} x)\Big|_{x=-1} = \frac{1}{2},$$

$$f'_-(-1) = \lim_{x \to -1^-} \frac{-\frac{\pi}{4} + \frac{x-1}{2} + \frac{\pi}{4}}{x+1} = +\infty.$$

Therefore, $f'(-1)$ does not exist.

4. (a) $\displaystyle\lim_{x \to a} \frac{xf(a) - af(x)}{x-a} = \lim_{x \to a} \frac{(x-a)f(a) - a(f(x) - f(a))}{x-a} = f(a) - af'(a).$

   (b) $\displaystyle\lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{x-a} = \lim_{x \to a} \frac{(f(x) - f(a))g(a) - f(a)(g(x) - f(a))}{x-a}$
   $= f'(a)g(a) - f(a)g'(a).$

   (c) For each $1 \le j \le k$,

$$\lim_{n \to \infty} n\left(f\left(a + \frac{j}{n}\right) - f(a)\right) = \lim_{n \to \infty} j \cdot \frac{f\left(a + j/n\right) - f(a)}{j/n} = j \cdot f'(a).$$

   Hence the required limit is $\displaystyle\sum_{j=1}^{k} j \cdot f'(a) = \frac{k(k+1)}{2} f'(a).$

5. For each $1 \le k \le 2019$,

$$\lim_{x \to 0} \frac{1}{x} f\left(\frac{x}{k}\right) = \frac{1}{k} \lim_{x \to 0} \frac{k}{x}\left(f\left(\frac{x}{k}\right) - f(0)\right) = \frac{1}{k} f'(0).$$

Hence,

$$\lim_{x \to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \cdots + f\left(\frac{x}{2019}\right)\right) = f'(0)\left(1 + \frac{1}{2} + \cdots + \frac{1}{2019}\right).$$

6. First note that

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \ell \implies \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \frac{f(x)}{x^2} = 0 \cdot \ell = 0.$$

Since f is continuous at 0, we must have $f(0) = 0$. Hence,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{f(x)}{x^2} = 0 \cdot \ell = 0.$$

Therefore, $f'(0)$ exists and equals 0.

7. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$. We start by observing that

$$a_1 + 2a_2 + \cdots + na_n = f'(0).$$

It is given that $|f(x)| \leq |\sin x|$ for all $x \in \mathbb{R}$ and we know that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$. Hence, we can say that $|f(x)| \leq |x|$. Since $f(0) = 0$, we have

$$\left|\frac{f(x) - f(0)}{x}\right| \leq 1 \text{ for all } x \neq 0 \implies |f'(0)| = \lim_{x \to 0} \left|\frac{f(x) - f(0)}{x}\right| \leq 1.$$

8. (a) Write

$$\frac{g(f(x)) - g(f(a))}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a},$$

and the rest is easy.

(b) Split up into two cases, based on whether $f(x) = f(a)$ or not.

(c) Let $b = f(a)$. We have defined

$$Q(x) = \begin{cases} \dfrac{g(f(x)) - g(b)}{f(x) - b} & \text{if } f(x) \neq b, \\ g'(b) & \text{if } f(x) = b. \end{cases}$$

Fix any $\varepsilon > 0$. From the definition of $g'(b)$, we know that there exists $\delta > 0$ such that

$$0 < |y - b| < \delta \implies \left|\frac{g(y) - g(b)}{y - b} - g'(b)\right| < \varepsilon.$$

Hence, $|Q(x) - g'(b)| < \varepsilon$ holds whenever $0 < |f(x) - b| < \delta$. The advantage with Q is that even when $f(x) = b$, we have $|Q(x) - g'(b)| = 0 < \varepsilon$. Thus,

$$0 \leq |f(x) - b| < \delta \implies |Q(x) - g'(b)| < \varepsilon.$$

Now, f is continuous at $x = a$, and $f(a) = b$. Hence for the above $\delta > 0$, there exists $\delta' > 0$ such that

$$0 < |x - a| < \delta' \implies |f(x) - b| < \delta \implies |Q(x) - g'(b)| < \varepsilon.$$

This proves that $\lim_{x \to a} Q(x) = g'(b)$.

(d) In light of the identity $\dfrac{g(f(x)) - g(f(a))}{x - a} = Q(x) \cdot \dfrac{f(x) - f(a)}{x - a}$, the proof follows from part (c) above.

9. Take $f(x) = x^2$ and $g(x) = |x|$. Note that $g'(f(0))$ does not exist, but $(g \circ f)'(0)$ exists.

10. Define $Q(x)$ as in Problem 8 (above) and write

$$\frac{f(x) - f(a)}{x - a} = \frac{1}{Q(x)} \cdot \frac{g(f(x)) - g(f(a))}{x - a}. \tag{$\star$}$$

Now, $\lim_{x\to a} Q(x) = g'(f(a)) \neq 0$, so we can say that $\lim_{x\to a} \frac{1}{Q(x)} = \frac{1}{g'(f(a))}$. And since $g \circ f$ is differentiable at a, we have

$$\lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x\to a} \frac{(g\circ f)(x) - (g\circ f)(a)}{x-a} = (g\circ f)'(a).$$

Therefore, letting $x \to a$ in ($\star$), we get the desired result.

11. Use the last problem with g^{-1} in place of f, and d in place of a.

12. Define $g : [0,1] \to [0,1]$, $g(x) = x^2$. The inverse function would be $g^{-1}(y) = \sqrt{y}$, defined on same interval. Note that g is differentiable at 0, g^{-1} is continuous at 0, but $g'(0) = 0$. Also observe that g^{-1} is not differentiable at $g(0) = 0$.

13. The key idea is to take log. Doing that, and using continuity of $\log(\cdot)$, we have

$$\log \lim_{n\to\infty} \left(\frac{f\left(a + \frac{1}{n}\right)}{f(a)} \right)^n = \lim_{n\to\infty} n \log \left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)} \right) = \lim_{n\to\infty} \frac{\log f\left(a+\frac{1}{n}\right) - \log f(a)}{1/n}.$$

We recognize this limit to be the derivative of $\log f(x)$ at $x = a$. Hence the last expression equals $= (\log f)'(a) = f'(a)/f(a)$. Finally, using the continuity of $x \mapsto e^x$, we conclude that the desired limit exists and equals $e^{f'(a)/f(a)}$.

14. Again, the idea is to take log and consider the limit as the derivative of a function.[2]

$$\begin{aligned}\lim_{n\to\infty} n \log\tan\left(\frac{\pi}{4} + \frac{1}{n}\right) &= \lim_{n\to\infty} \frac{\log\tan\left(\frac{\pi}{4}+\frac{1}{n}\right) - \log\tan\frac{\pi}{4}}{1/n} \\ &= \lim_{n\to\infty} \frac{f(\pi/4 + 1/n) - f(\pi/4)}{1/n} \quad (\text{where } f(x) = \log\tan x) \\ &= \lim_{h\to 0} \frac{f(\pi/4+h) - f(\pi/4)}{h} \\ &= f'(\pi/4).\end{aligned}$$

Since $f'(x) = \sec^2 x/\tan x$, we have $f'(\pi/4) = 2$, and hence the required limit equals e^2.

15. Once again, the idea is to take log and consider the limit as the derivative of a function!

$$\log\left(\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{1/x} \right) = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

where $f(x) = \frac{1}{n}\sum_{j=1}^n a_j^x$. Since $f'(x) = \frac{1}{n}\sum_{j=1}^n a_j^x \log a_j$, the above limit equals $f'(0) = \frac{1}{n}\sum_{j=1}^n \log a_j = \log((a_1 a_2 \cdots a_n)^{1/n})$. Therefore, the desired limit equals $(a_1 a_2 \cdots a_n)^{1/n}$.

[2]Personally, I found this idea to be quite an useful tool for solving problems.

16. (a) For $n > 0$, use $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})$. And for $n < 0$, substitute $m = -n$.

(b) Substitute $y = x^{1/n}$ and $b = a^{1/n}$. Note that $x \to a \iff y \to b$. Hence,

$$\lim_{x \to a} \frac{x^{1/n} - a^{1/n}}{x - a} = \lim_{y \to b} \frac{y - b}{y^n - b^n} = \frac{1}{nb^{n-1}} = \frac{1}{n} a^{\frac{1}{n}-1}.$$

Alternately, you may use the *Inverse Function Theorem.* The function $x^{1/n}$ is the inverse of the function x^n (here x is restricted to take positive values only).

(c) You may combine the parts (a) and (b) using the chain rule. Alternately, for $r = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, you can substitute $y = x^{1/q}$ and $b = a^{1/q}$.

17. Write $f(x) = x^{\sqrt{3}} = e^{\sqrt{3}\log x}$. Using the chain rule we get $f'(x) = e^{\sqrt{3}\log x} \cdot \frac{\sqrt{3}}{x} = \sqrt{3}x^{\sqrt{3}-1}$.

18. If we assume that $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$, then

$$\begin{aligned}
\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{2h} + \frac{f(c) - f(c-h)}{2h} \right) \\
&= \left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{2h} \right) + \left(\lim_{h \to 0} \frac{f(c-h) - f(c)}{2(-h)} \right) \\
&= \frac{f'(c)}{2} + \frac{f'(c)}{2} = f'(c).
\end{aligned}$$

However, the existence of this limit does not imply that $f'(c)$ exists. For instance, take $c = 0$ and $f(x) = |x|$. In fact, when $c = 0$, the limit exists for any even function (which need not be differentiable at 0).

19. *Intuition:* $\frac{f(z_n)-f(x_n)}{z_n-x_n}$ *is the slope of the line that connects* $(x_n, f(x_n))$ *and* $(z_n, f(z_n))$*. As* $n \to \infty$*, both the endpoints come closer and closer to* a*. The result (to be shown) says that slope of that chord converges to the slope of the tangent at* a*. What the definition says is that*

$$f'(a) = \lim_{n \to \infty} \frac{f(z_n) - f(a)}{z_n - a} = \lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a}.$$

However, the slope $\frac{f(z_n)-f(x_n)}{z_n-x_n}$ *is of a different nature.*

Let us start with $\dfrac{f(z_n) - f(x_n)}{z_n - x_n} = \dfrac{f(z_n) - f(a)}{z_n - x_n} + \dfrac{f(a) - f(x_n)}{z_n - x_n}$. We can rewrite it as

$$\underbrace{\frac{f(z_n) - f(x_n)}{z_n - x_n}}_{\text{Call it } A_n} = \underbrace{\frac{f(z_n) - f(a)}{z_n - a}}_{\text{Call it } B_n} \cdot \underbrace{\frac{z_n - a}{z_n - x_n}}_{\text{Call it } \lambda_n} + \underbrace{\frac{f(x_n) - f(a)}{x_n - a}}_{\text{Call it } C_n} \cdot \underbrace{\frac{a - x_n}{z_n - x_n}}_{\text{Its } (1-\lambda_n)}. \qquad (*)$$

Note, $0 < \lambda_n = \dfrac{z_n - a}{z_n - x_n} < 1$ and $\dfrac{z_n - a}{z_n - x_n} = 1 - \lambda_n$ for all $n \geq 1$. Therefore $(*)$ says that

$$\text{for every } n \geq 1,\ A_n \text{ lies } \underline{\text{between}}\ B_n \text{ and } C_n.$$

Now, as we mentioned in the beginning of the solution,

$$\lim_{n\to\infty} B_n = f'(a) = \lim_{n\to\infty} C_n.$$

Hence applying the idea of Sandwich theorem, we can conclude that $\lim_{n\to\infty} A_n = f'(a)$.

Remark. The last argument can be made precise as follows. Take any $\varepsilon > 0$. Since $\lim_{n\to\infty} B_n = f'(a) = \lim_{n\to\infty} C_n$, there exists N such that, for every $n \geq N$, we have

$$f'(a) - \varepsilon < B_n < f'(a) + \varepsilon, \text{ as well as } f'(a) - \varepsilon < C_n < f'(a) + \varepsilon.$$

And we know that for each $n \geq 1$, A_n lies <u>between</u> B_n and C_n. Therefore,

$$f'(a) - \varepsilon < A_n < f'(a) + \varepsilon \text{ for every } n \geq N.$$

Hence we conclude that $\lim_{n\to\infty} A_n$ must exist and equal to $f'(a)$.

20. No, it is not necessary. However, finding a counter-example for this problem is very tricky. Notice that the assumptions in this problem differ from the previous problem in one place, namely that here we can take $0 < x_n < z_n$. Define $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. We know that $f'(0) = 0$. Take

$$x_n = \frac{1}{2n\pi + \pi/2} = \frac{2}{(4n+1)\pi}, \text{ and } z_n = \frac{1}{2n\pi}.$$

Observe that $0 < x_n < z_n$ and $f(x_n) = x_n^2$ and $f(z_n) = 0$, for each $n \geq 1$. Hence,

$$\frac{f(z_n) - f(x_n)}{z_n - x_n} = \frac{-x_n^2}{z_n - x_n} = \ldots = -\frac{2}{\pi}\frac{4n}{4n+1}$$

Therefore

$$\lim_{n\to\infty} \frac{f(z_n) - f(x_n)}{z_n - x_n} = -\frac{2}{\pi}$$

which is not same as $f'(0)$.

Comment. Does it contradict your intuition? It probably does, but it is what it is.

21. One such function is $f(x) = |x-1| + |x-2| + \cdots + |x-5|$.

22. First show that f is discontinuous everywhere except at $x = 1$. Hence $f'(a)$ does not exist

when $a \neq 1$. To see whether f is differentiable at $x = 1$, note that

$$\frac{f(x) - f(1)}{x - 1} = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \text{ is rational} \\ \frac{2x-1-1}{x-1} & \text{if } x \text{ is irrational} \end{cases} = \begin{cases} x+1 & \text{if } x \text{ is rational} \\ 2 & \text{if } x \text{ is irrational} \end{cases}.$$

Since $\lim_{x\to 1}(x+1) = 2$, it follows that $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 2$, implying that $f'(1) = 2$.

Remark. We can show (in a similar fashion) that if f is defined as

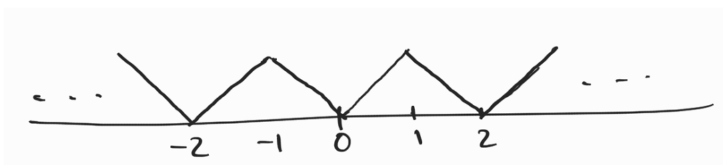
$$f(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{Q} \\ h(x) & \text{if } x \notin \mathbb{Q} \end{cases}$$

then f is continuous at exactly the points where $g(x) = h(x)$ holds, and differentiable at exactly the points where both $g(x) = h(x)$ and $g'(x) = h'(x)$ holds.

23. The function $f(x) = \begin{cases} x^2(x-1)^2(x-2)^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is differentiable only at $x = 0, 1, 2$.

Challenge: Can you find a function which is continuous only at $x = 0, 1$ and differentiable only at $x = 0$?

24. One example is the following function: $f(x) = |x - 2n|$ for $2n - 1 \leq x < 2n + 1$ where $n \in \mathbb{Z}$. Construction of this function is actually very easy if we think using the graph of the function, which is shown below. Note that this function can also be written in a more



compact form: $f(x) = \sin^{-1}|\sin(\frac{\pi}{2}x)|$, $x \in \mathbb{R}$. Another good example is $\lfloor x \rfloor + \{x\}^2$ where $\lfloor x \rfloor$ is the floor function, and $\{x\}$ is the fractional part of x, defined as $x - \lfloor x \rfloor$. All of these examples were suggested my students (unfortunately, I forgot exactly who suggested what).

25. One such function is $f(x) = \begin{cases} \sin^2(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$. More generally, any function f of the form

$$f(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

would work where g is a function which vanishes exactly at the integers and there it has a zero derivative. (That is, $g(a) = 0 \iff a \in \mathbb{Z}$ and $g'(a) = 0$ for every $a \in \mathbb{Z}$.) Relate this with the remark given right after the solution of problem 22 above.

26. Let $W(x)$ be a function which is continuous everywhere on $\mathbb{R}$, but nowhere differentiable. Define $\widetilde{W}(x) = (\sin \pi x)W(x)$, as given in the question. It follows immediately that $\widetilde{W}(x)$ is continuous on $\mathbb{R}$. Observe that we can write

$$\frac{\widetilde{W}(x) - \widetilde{W}(a)}{x-a} = \frac{(\sin \pi x)W(x) - (\sin \pi a)W(a)}{x-a}$$
$$= \frac{\sin \pi x - \sin \pi a}{x-a} \cdot W(x) + \frac{W(x) - W(a)}{x-a} \cdot \sin \pi a. \qquad (\heartsuit)$$

Now when we let $x \to a$, for the first summand (colored blue) does have a limit (since $W(x)$ is continuous at $x = a$), namely $\pi \cos(\pi a)W(a)$. But the part colored red does not have a limit (since $W(x)$ is not differentiable at $x = a$). Luckily, the green part saves us when, and only when, $a \in \mathbb{Z}$. Using ($\heartsuit$) we can easily show that

$$\lim_{x\to a} \frac{\widetilde{W}(x) - \widetilde{W}(a)}{x-a} = \begin{cases} \pi\cos(\pi a)W(a) & \text{if } a \in \mathbb{Z} \\ \text{does not exist} & \text{if } a \notin \mathbb{Z} \end{cases}.$$

Note that in the above proof we did not require any particular form of $W(x)$. The choice of $\sin(\pi x)$ is also optional: instead of $\sin(\pi x)$ we could have used any function which is differentiable on $\mathbb{R}$, and vanishes at $x = a$ if and only if $a \in \mathbb{Z}$.

Remark. Let $W(x)$ be as above. If $f(x)$ be any function differentiable at every $x \in \mathbb{R}$, then the function $f(x)W(x)$ is continuous everywhere but differentiable at exactly the points where f vanishes!

27. Use induction on n and the following identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

28. Use the previous problem (i.e. use Leibnitz rule, with $f(x) = 1/x$ and $g(x) = \log x$). Alternative solution: guess a form of the n-th derivative at x and prove it using induction. Why do we need to find derivative at any value of x? Because differentiating $f^{(n)}(1)$ w.r.t. x results in 0, not $f^{(n+1)}(1)$.

29. No. From $f(x) > g(x)$ one can not tell how the rate of growth of $f(x)$ relates to that of $g(x)$. Although $f > g$, it might be the case that f is decreasing and g is increasing. e.g. $f(x) = e^{-x}, g(x) = -e^{-x}$. Another type of counter-example would be: take g to be any differentiable function and take $f(x) = g(x) + 1/x^2$.

30. Yes, both are possible. Consider the following counter-examples.

(a) f is unbounded on I, but f' is bounded: take $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$.

(b) f is bounded on I, but f' is unbounded: take $f : (0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$.

Remark. If I is a bounded interval, and f' is bounded on I, then it actually implies that f must also be bounded on I. We would be able to prove this easily once we learn integration.

31. First observe that,

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \implies P'(x) = P(x) \left(\frac{1}{x - r_1} + \cdots + \frac{1}{x - r_n} \right).$$

Hence,

$$P'(r_k) = \prod_{1 \le i \le n, i \ne k} (r_k - r_i).$$

This is because the expression for P' has n terms, all but one of which have $(x - r_k)$ as a factor, and the only term not having this property is

$$(x - r_1) \cdots (x - r_{k-1})(x - r_{k+1}) \cdots (x - r_n) = \prod_{1 \le i \le n, i \ne k} (x - r_i).$$

Now, the identity to be shown is the following

$$Q(x) = \sum_{k=1}^{n} \frac{Q(r_k)}{P'(r_k)} \frac{P(x)}{(x - r_k)}.$$

Plugging in the value of $P'(r_k)$ for each k, we can rewrite it as

$$Q(x) = \sum_{k=1}^{n} \left(Q(r_k) \prod_{1 \le i \le n, i \ne k} \frac{(x - r_i)}{(r_k - r_i)} \right). \tag{†}$$

To prove the above, note that each side of the equation (†) is a polynomial of degree $< n$. So, it suffices to show that these two polynomials have the same value at n points. Note that when we put $x = r_k$ in the RHS of (†), all but one of the summands vanish. And the only summand that does not vanish equals $Q(r_k)$. Therefore, the RHS and LHS match at $x = r_1, r_2, \ldots, r_n$. This completes the proof.

Remark. The equation (†) is known as Lagrange's Interpolation formula. Suppose $Q(x)$ is a polynomial of any degree (deg Q maybe more than n). If we are given the value of $Q(x)$ at n points, $r_1, \ldots, r_n$, we can approximate $Q(x)$ using the $(n - 1)$ degree polynomial in the RHS of (†).