Derivatives

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$f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ and let $a \in D$ be a limit point.

(So that $\lim\limits_{x \to a}$ makes sense)

We say that f is differentiable at $x = a$, with derivative l, if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = l.$$

We denote l by $f'(a)$ or $\left.\frac{d}{dx} f(x)\right|_{x=a}$.

Properties

① If f and g are both diff'ble at $x = a$, then so will be $f + g$ and $f - g$, with

$$(f \pm g)'(a) = f'(a) \pm g'(a).$$

(2) If $f'(a)$ exists, then f will
(means f is diff'ble at $x=a$)
be cont. at $x=a$ as well.

Why? We have $l \in \mathbb{R}$ s.t.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = l.$$

Hence,

$$\lim_{x \to a} f(x) - f(a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a)$$

$$= l \times 0 = 0.$$

Therefore, $\lim_{x \to a} f(x) = f(a)$,

i.e., f is cont. at $x = a$.

diff'ble at a $\overset{\Longrightarrow}{\not\Longleftarrow}$ cont. at a

e.g., $f(x) = |x|$ is cont. at $x=0$,

but not diff'ble at $x=0$ (why?).

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \underbrace{\lim_{x\to 0} \frac{|x|}{x}}_{\text{does not exist}}$$

③ f, g diff'ble at $x=a$, then so is fg, with

$$\frac{d}{dx} f(x)\, g(x) \Big|_{x=a} = f(a)\, g'(a) + f'(a)\, g(a).$$

$$\frac{f(x)\, g(x) - f(a)\, g(a)}{x-a}$$

$$= \underbrace{f(x)}_{\downarrow\, f(a)} \underbrace{\frac{g(x)-g(a)}{x-a}}_{\to\, g'(a)} + g(a) \underbrace{\frac{f(x)-f(a)}{x-a}}_{\to\, f'(a)}$$

④ If f, g both diff'ble at $x=a$, and $g(a) \neq 0$, then f/g will also be diff'ble at $x=a$, and

$$\left(f/g\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}.$$

$$\left(\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}\right) / (x-a)$$

$$= \frac{g(a)\dfrac{(f(x) - f(a))}{x-a} - f(a)\dfrac{g(x)-g(a)}{x-a}}{g(x)\, g(a)}$$

$$\rightarrow \frac{g(a)\, f'(a) - f(a)\, g'(a)}{g(a)^2}.$$

(as $x \to a$)

$$\frac{d}{dx}(x^n) = n x^{n-1}, \quad n \in \mathbb{N}$$

$$\frac{d}{dx}(x^n) = n x^{n-1}, \quad n \in \mathbb{Z}, \quad x \neq 0.$$

Does this also hold for $n \in \mathbb{Q}$?

Suppose $f(x) = x^{p/q}$, $x > 0$.

$$\lim_{x \to a} \frac{x^{p/q} - a^{p/q}}{x - a}$$

Assume $q \in \mathbb{N}$, $p \in \mathbb{Z}$.

$$y = x^{1/q}, \quad b = a^{1/q}$$

$$x \to a \Rightarrow y \to b.$$

$$= \lim_{y \to b} \frac{y^p - b^p / y - b}{y^q - b^q / y - b}$$

$$= p b^{p-1} / q b^{q-1} = \frac{p}{q} a^{p/q - 1}.$$

(check)

Next, is it true that

$$\frac{d}{dx}(x^r) = r x^{r-1}$$

for any $r \in \mathbb{R}$? Yes, we'll see.

Take q_n seq. of rationals, $q_n \to r$

$$\lim_{x\to a} \frac{x^{q_n} - a^{q_n}}{x-a} = q_n a^{q_n - 1}$$

$$\underline{\lim_{n\to\infty} \lim_{x\to a} \frac{x^{q_n} - a^{q_n}}{x-a} = r a^{r-1}.}$$

why are they same?

$$\lim_{x\to a} \frac{x^{r} - a^{r}}{x-a}$$

$$\underline{= \lim_{x\to a} \lim_{n\to\infty} \frac{x^{q_n} - a^{q_n}}{x-a}.}$$

We'll prove $\frac{d}{dx}(x^r) = r x^{r-1}$ in a different manner.

5 (<u>Chain-rule</u>)

Suppose that f diff'ble at $x=a$, g diff'ble at $y=f(a)$, and $g\circ f(x) = g(f(x))$ is well-defined.

Then, $(g\circ f)'(a) = g'(f(a))\, f'(a)$.

$$\frac{g(f(x)) - g(f(a))}{x - a}$$

$$= \underbrace{\frac{g(f(x)) - g(f(a))}{f(x) - f(a)}}_{\substack{\to g'(f(a)) \\ \text{as } x \to a}} \times \underbrace{\frac{f(x) - f(a)}{x - a}}_{\substack{\to f'(a) \\ \text{as } x \to a}}$$

Define

$$Q(x) = \begin{cases} \dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)} & \text{if } f(x) \neq f(a) \\ g'(f(a)) & \text{if } f(x) = f(a) \end{cases}$$

Call $b = f(a)$.

$$\lim_{y \to b} \frac{g(y) - g(b)}{y - b} = g'(b).$$

$\therefore \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |y - b| < \delta$ implies

$$\left| \frac{g(y) - g(b)}{y - b} - g'(b) \right| < \varepsilon.$$

And f is cont. at $x = a$.

So, given $\delta > 0$, $\exists\ \eta > 0$ such that

$0 < |x - a| < \eta \Rightarrow |f(x) - f(a)| < \delta$.

So, if $0 < |x - a| < \eta$ such that

$\underline{0 <} \, |\underbrace{f(x)}_{=y} - \underbrace{f(a)}_{=b}| < \delta$, then

$$\left| \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} - g'(f(a)) \right| < \varepsilon,$$

i.e., $|Q(x) - g'(f(a))| < \varepsilon$.

If $0 < |x - a| < \eta$ is such that

$f(x) = f(a)$, then

$$|Q(x) - g'(f(a))| = 0 < \varepsilon.$$

So, in either case, $0 < |x - a| < \eta$

implies $|Q(x) - g'(f(a))| < \varepsilon$.

Therefore,

$$\lim_{x\to a} Q(x) = g'(f(a)).$$

Why bother about $Q(x)$? Because,

$$\frac{g(f(x)) - g(f(a))}{x - a}$$

$$= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \times \frac{f(x) - f(a)}{x - a}$$

this is not valid when $f(x) = f(a)$,

but we can always write

$$\frac{g(f(x)) - g(f(a))}{x - a} = Q(x) \cdot \frac{f(x) - f(a)}{x - a},$$

regardless of whether $f(x) \neq f(a)$.

letting $x \to a$, the RHS converges to

$g'(f(a))\, f'(a)$. Done!

$(e^x)' = e^x, \quad (\log x)' = \frac{1}{x}$

Applⁿ of chain rule

- $\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{x \log a}\right)$

(Chain rule) $= e^{x \log a} \frac{d}{dx}(x \log a)$

$= a^x \log a.$

- $\frac{d}{dx}(x^r) = \frac{d}{dx}\left(e^{r \log x}\right)$

$= e^{r \log x} \frac{d}{dx}(r \log x)$

$= x^r \cdot \frac{r}{x}$

$= r x^{r-1}!$

$(\sin x)' = \cos x \qquad (\tan x)' = \sec^2 x$

$(\cos x)' = -\sin x \qquad$ etc.

Q. Find $\lim\limits_{n\to\infty} \tan^n\left(\frac{\pi}{4}+\frac{1}{n}\right)$.

As $n\to\infty$, $\tan\left(\frac{\pi}{4}+\frac{1}{n}\right)\to 1$ $\quad 1^\infty$

Take log

$$\lim_{n\to\infty} \log \tan^n\left(\frac{\pi}{4}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} n \log \tan\left(\frac{\pi}{4}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\log\tan\left(\frac{\pi}{4}+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$f(x) = \log\tan x$$

$$f(\pi/4) = \log 1 = 0$$

$$= \lim_{n\to\infty} \frac{f\left(\frac{\pi}{4}+\frac{1}{n}\right) - f\left(\frac{\pi}{4}\right)}{\frac{1}{n}}$$

$$= \lim_{h\to 0} \frac{f(\pi/4+h) - f\left(\frac{\pi}{4}\right)}{h}$$

$$= f'\left(\frac{\pi}{4}\right).$$

$$f(x) = \log \tan x$$

$$f'(x) = \frac{1}{\tan x} \sec^2 x.$$

$$f'(\pi/4) = \frac{\sec^2 \pi/4}{\tan \pi/4} = 2.$$

Therefore,

$$\lim_{n\to\infty} \log \tan^n\left(\frac{\pi}{4} + \frac{1}{n}\right) = 2.$$

$$\Rightarrow \lim_{n\to\infty} \tan^n\left(\frac{\pi}{4} + \frac{1}{n}\right) = e^2.$$

(Ans)

$(\because x \mapsto e^x$ is cont.)

Q. Given $a_1, a_2, \ldots, a_n > 0$, Calculate the following limit:

$$\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x}.$$

$$\lim_{x\to 0} \frac{\log\left(\frac{1}{n}\sum_{i=1}^{n} a_i^x\right)}{x}$$

$$\text{Call } f(x) = \log\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} a_i^x\right).$$

$$f(0) = \log\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} 1\right) = 0.$$

$$= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= f'(0) \qquad f(x) = \log\left(\frac{1}{n}\sum_{i=1}^{n} a_i^x\right)$$

$$f'(x) = \frac{1}{\sum_{i=1}^{n} a_i^x / n} \times \frac{1}{n}\sum_{i=1}^{n} a_i^x \log a_i$$

$$f'(0) = \frac{1}{n}\sum_{i=1}^{n} \log a_i$$

$$= \log G, \quad G = (a_1 a_2 \cdots a_n)^{1/n}.$$

So, $$\lim_{x\to 0} \log\left(\left(\frac{1}{n}\sum_{i=1}^{n} a_i^x\right)^{1/x}\right) = \log G$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{1}{n}\sum_{i=1}^{n} a_i^x\right)^{1/x} = G. \quad \text{(Ans)}$$

Q. Suppose that $a_1, a_2, \ldots, a_n$ are n real numbers such that

$$\left|\sum_{j=1}^{n} a_j \sin(jx)\right| \leq |\sin x|$$

for every $x \in \mathbb{R}$. Show that,

$$|a_1 + 2a_2 + \ldots + na_n| \leq 1.$$

Sol^n: Let us define

$$f(x) = \sum_{k=1}^{n} a_k \sin(kx), \ x \in \mathbb{R}.$$

We are given that

$$|f(x)| \leq |\sin x| \qquad (*)$$

for every $x \in \mathbb{R}$. Also note that,

$$f'(x) = \sum_{k=1}^{n} k a_k \cos(kx).$$

Thus, $\sum_{k=1}^{n} k a_k = f'(0)$.

Here $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$. $\quad f(0) = 0$

Therefore,

$$|f'(0)| = \left| \lim_{x\to 0} \frac{f(x)}{x} \right|$$

$$= \lim_{x\to 0} \frac{|f(x)|}{|x|} \quad \text{[Since } |\cdot| \text{ is cont.]}$$

$$\leq \lim_{x\to 0} \frac{|\sin x|}{|x|} \quad \text{[from } (*)\text{]}$$

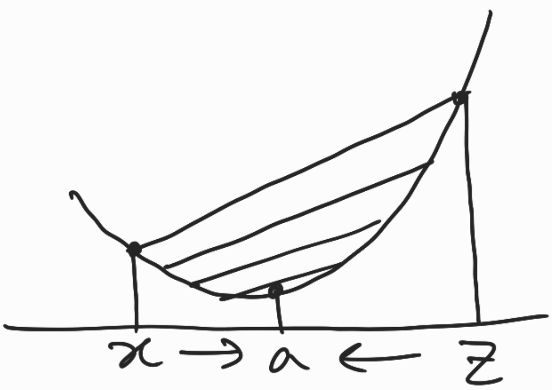
$$= \left| \lim_{x\to 0} \frac{\sin x}{x} \right|$$

$$= 1.$$

Slope of the chord

$$\frac{f(x) - f(a)}{x - a}$$

As $x \to a$, limiting slope is $f'(a)$.



$$\frac{f(z) - f(x)}{z - x} \xrightarrow{?} f'(a)$$

$$x < a < z.$$

Q. Suppose that $(x_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ are two sequences such that

$$x_n < a < z_n \text{ for all } n \geq 1,$$

and $x_n \to a$, $z_n \to a$ as $n \to \infty$.

Assuming $f'(a)$ exists, show that

$$\lim_{n\to\infty} \frac{f(z_n) - f(x_n)}{z_n - x_n} = f'(a).$$

Sol^n: Since $z_n \to a$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{f(z_n) - f(a)}{z_n - a} = f'(a).$$

Similarly, from $x_n \to a$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{f(a) - f(x_n)}{a - x_n} = f'(a).$$

Let us call,

$$A_n = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \quad B_n = \frac{f(z_n) - f(a)}{z_n - a},$$

and,

$$C_n = \frac{f(a) - f(x_n)}{a - x_n}.$$

Then

$$A_n = \frac{z_n - a}{z_n - x_n} B_n + \frac{a - x_n}{z_n - x_n} C_n.$$

$$\lambda_n = \frac{z_n - a}{z_n - x_n} \in (0, 1).$$

$(\because x_n < a < z_n)$

$$A_n = \lambda_n B_n + (1 - \lambda_n) C_n.$$

Now, we know that

$$\lim_{n \to \infty} B_n = f'(a) = \lim_{n \to \infty} C_n.$$

Since A_n is sandwiched between B_n and C_n, $\lim_{n \to \infty} A_n = f'(a)$.

Why?

Fix any $\varepsilon > 0$. Then $\exists N$ s.t. both B_n and C_n lies in the ε-nbd. of $f'(a)$ for $n \geq N$.

Since $A_n = \underbrace{\lambda_n B_n + (1-\lambda_n) C_n}_{\text{(weighted mean)}}$,

we can say that A_n lies between B_n and C_n and hence also belongs to the ε-nbd of $f'(a)$, for every $n \geq N$.

This completes the proof.