Solution-sketches

1. (a) Consider $h(x) = e^{-x}g(x)$ and show that h is strictly increasing. Hence conclude from $h(x_0) = 0$ that $h(x) > 0$ for $x > x_0$ and $h(x) < 0$ for $x < x_0$.

   (b) Define $g(x) = ae^x - (1 + x + x^2/2)$ and observe that

   $$g'(x) < g(x) \text{ for every } x \in \mathbb{R}.$$

   Now, the equation $g(x) = 0$ must have at least one real root, since

   $$\lim_{x\to\infty} g(x) = +\infty, \quad \lim_{x\to-\infty} g(x) = -\infty$$

   and g is continuous. If x_0 be one solution of the equation $g(x) = 0$, we can apply the result proved in part (a) to conclude that $g(x) > 0$ for $x > x_0$ and $x < x_0$.

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2. (a) For $\alpha \geq 0$ define

   $$f(\alpha) = (1-\alpha)H(\alpha) = \log\left(\sum_{i=1}^{n} p_i^\alpha\right).$$

   Then

   $$\lim_{\alpha\to 1} H(\alpha) = -\lim_{\alpha\to 1} \frac{f(\alpha) - f(1)}{\alpha - 1} = f'(1) = -\sum_{i=1}^{n} p_i \log p_i.$$

   (b) Straightforward calculation leads to the following

   $$(1-\alpha)^2 H'(\alpha) = \log\left(\sum_{i=1}^{n} p_i^\alpha\right) + (1-\alpha)\sum_{i=1}^{n} z_i \log p_i.$$

   Now multiply the first term on the above RHS with $\sum_{i=1}^n z_i = 1$ to get

   $$(1-\alpha)^2 H'(\alpha) = \sum_{i=1}^{n} z_i \log p_i - \sum_{i=1}^{n} z_i \log\left(\frac{p_i^\alpha}{\sum_{i=1}^n p_i^\alpha}\right) = \sum_{i=1}^{n} z_i \log(p_i/z_i).$$

   (c) We know, $\log x < x - 1$ for $x > 0$. Hence

   $$(1-\alpha)^2 H'(\alpha) = \sum_{i=1}^{n} z_i \log(p_i/z_i) \leq \sum_{i=1}^{n} z_i(p_i/z_i - 1) = 1 - 1 = 0.$$

   This holds for every $\alpha \geq 0, \alpha \neq 1$, so we can say that H is decreasing (non-increasing) on $[0, 1)$ and $(1, \infty)$. We can define $H(1) = \lim_{\alpha\to 1} H(\alpha)$ (which exists from part (a)) and use a limiting argument (e.g., take $x < z < 1$ and let $z \to 1$) to show that $H(x) \geq H(1) \geq H(y)$ for every $x < 1 < y$.

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3. If we can find the minimum distance of any point on the curves from the origin, that would be radius of the largest circle that can be inscribed in the given region. By symmetry, it suffices to work only in the first quadrant. Hence our goal is to minimize

$$\sqrt{x^2 + f(x)^2} = \sqrt{x^2 + \frac{1}{(1+x^2)^2}}, \ x \geq 0,$$

or equivalently, its square, which we denote by $g(x)$ (say). A little thought reveals that it suffices (can you see why?) to minimize instead

$$h(x) = x + \frac{1}{(1+x)^2}, \ x \geq 0.$$

Now show (using derivatives) that h is minimized at $x = 2^{1/3} - 1 = a$ (say). Then $g(x) = h(x^2)$ will be minimized at $x = \sqrt{a}$ (can you see why?). Therefore the required area is given by

$$\pi g(\sqrt{a}) = \pi h(a) = \pi \left(a + (1+a)^{-2}\right) = \pi \left(2^{1/3} - 1 + 2^{-2/3}\right).$$

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4. Let, if possible, $S = \{x \in [0, \pi] : f(x) = 0\}$ be an infinite set. Then we can pick a sequence $(x_n)_{n \geq 1}$ consisting of distinct numbers such that $x_n \in S$ for each $n \geq 1$. Since $0 \leq x_n \leq \pi$, we can pick a convergent subsequence, say $(y_k)_{k \geq 1} = (x_{n_k})_{k \geq 1}$. Suppose that it converge to y. Since $0 \leq y \leq \pi$, f is continuous at y, and so

$$f(y) = \lim_{n \to \infty} f(y_n) = 0.$$

Finally, conclude using $f(y) = 0 = f(y_n)$ for all $n \geq 1$ that

$$f'(y) = 0.$$

This contradicts the given fact that there is no $t \in [0, \pi]$ such that $f(t) = f'(t) = 0$.

Comment: If $y_n = y$ for every $n \geq 1$, then $\frac{f(y_n) - f(y)}{y_n - y}$ would be undefined for every $n \geq 1$. Such cases are avoided by our assumption that x_n's are all distinct.

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