Ramanujan School of Mathematics

Class Test on Calculus

Sept 2019

Total marks: $10 \times 5 = 50$ Time: 2 hours.

Attempt all the questions. Answers without proper explanations will fetch zero. Show all your rough work – partial solutions may be rewarded. You can use any theorem/result without proving it again; but you have to state it properly.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x) \neq x$ for every $x \in \mathbb{R}$. Is it possible that there exists some $c \in \mathbb{R}$ such that $f(f(c)) = c$?

2. Let $f : [0, 1] \to \mathbb{R}$ be a function satisfying $f(2x) = 3f(x)$ for every $0 \leq x \leq 1/2$. If f is bounded, show that $\lim_{x \to 0+} f(x) = f(0)$.

3. Determine, with proof, whether the following statements are true or false: (If true then provide a proof, else provide a counter-example)
   (a) If $\lim_{x \to 0} f(x) = c$ then $\lim_{x \to 0} f(\sin x) = c$.
   (b) If $\lim_{x \to 0} f(\sin x) = c$ then $\lim_{x \to 0} f(x) = c$.

4. Determine, with proof, the value of the following limit
$$\lim_{n \to \infty} \tan^n \left(\frac{\pi}{4} + \frac{1}{n} \right).$$

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that given any two points $x_1 < x_2$, there exists a point x_3 between x_1 and x_2 such that $f(x_3) = g(x_3)$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

Do not cheat to yourself. All the best!

Teacher: Aditya Ghosh

Solutions

1. If $f(x)-x$ changes its sign then by continuity there exists a point x_0 such that $f(x_0)=x_0$. But since that is not allowed, we deduce that $f(x)-x$ is either positive for all x, or negative for all x. When $f(x)>x$ for all x, we find that

$$f(f(x)) > f(x) > x.$$

And when $f(x)<x$ for all x, we find that

$$f(f(x)) < f(x) < x.$$

Thus in both the cases, we cannot have a solution to the equation $f(f(x))=x$.

2. We are given that $f(2x)=3f(x)$ for every $x\in[0,1/2]$. This yields $f(0)=0$. Also deduce that for any natural number n,

$$f(2^n x) = 3^n f(x) \text{ for every } x\in[0,1/2^n]. \qquad (*)$$

Since f is bounded, we can assume that

$$|f(x)| \le M \text{ for every } x\in[0,1]$$

for some fixed $M>0$. Now, let $(x_n)_{n\ge 1}$ be a sequence in $[0,1]$ such that $x_n\to 0^+$. Our goal is to show that for **any** such sequence, $f(x_n)\to f(0)=0$. Given any $\varepsilon>0$, we take $k\in\mathbb{N}$ (sufficiently large) such that

$$M/3^k < \varepsilon.$$

Since $x_n\to 0^+$, for all sufficiently large n, say for $n\ge N$, it holds that

$$0\le x_n \le 1/2^k.$$

It follows from $(*)$ that for any $n\ge N$,

$$|f(x_n)| = \frac{|f(2^k x_n)|}{3^k} \le \frac{M}{3^k} < \varepsilon.$$

This implies, from the ε-definition of limit, that $f(x_n)\to 0$ as $n\to\infty$.

(Note, a crucial part of the solution is to first choose k and then choose N.)

3. Both are true. We use the result that $|\sin x|\le|x|$ for every $x\in\mathbb{R}$.

   (a) If $\lim_{x\to 0} f(x)=c$, then for any $\varepsilon>0$ there exists $\delta>0$ such that $0<|x|<\delta$ implies $|f(x)-c|<\varepsilon$. Since $|\sin x|\le|x|$, we can see that $|x|<\delta \implies |\sin x|<\delta$ which further implies $|f(\sin x)-c|<\varepsilon$.

(b) If $\lim_{x\to 0} f(\sin x) = c$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(\sin x) - c| < \varepsilon$. We may assume w.l.o.g. that $\delta < \pi/2$. The idea is to put $y = \sin x$, i.e., $x = \sin^{-1} y$. Take $\delta_1 = \sin\delta \in (0,1)$. Since $\sin(\cdot)$ is strictly increasing on $(-\pi/2, \pi/2)$, for $y = \sin x$ we can say that $0 < |\sin x| < \sin\delta \implies 0 < |x| < \delta$ which further implies that $|f(\sin x) - c| < \varepsilon$. Thus, for $0 < |y| < \delta_1 = \sin\delta$ we have $|f(y) - c| < \varepsilon$.

4. The key idea is to take logarithm (with base e, of course!). Let us denote

$$A_n = \tan^n\left(\frac{\pi}{4} + \frac{1}{n}\right), \ n \geq 1.$$

Observe that

$$\begin{aligned}
\lim_{n\to\infty} \log A_n &= \lim_{n\to\infty} n \log\tan\left(\frac{\pi}{4} + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} n \log\left(\frac{1 + \tan\frac{1}{n}}{1 - \tan\frac{1}{n}}\right) \\
&= \lim_{n\to\infty} n \log\left(1 + \frac{2\tan\frac{1}{n}}{1 - \tan\frac{1}{n}}\right) \\
&= \lim_{n\to\infty} n \frac{2\tan\frac{1}{n}}{1 - \tan\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{2\tan\frac{1}{n}}{\frac{1}{n}} = 2.
\end{aligned}$$

Since $x \mapsto e^x$ is a continuous map, the desired limit is

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} e^{\log A_n} = e^2.$$

5. Fix any $y \in \mathbb{R}$. For any natural number $n \geq 1$ we can use the given property with $x_1 = y - \frac{1}{n}$ and $x_2 = y + \frac{1}{n}$ to say that there exists $y_n \in [y - \frac{1}{n}, y + \frac{1}{n}]$ such that $f(y_n) = g(y_n)$. Since

$$y - \frac{1}{n} \leq y_n \leq y + \frac{1}{n} \text{ for every } n \geq 1,$$

Sandwich applies and tells us that $y_n \to y$ as $n \to \infty$. But we have

$$f(y_n) = g(y_n) \text{ for every } n \geq 1.$$

Letting $n \to \infty$ here, and using the continuity of f and g we can conclude that

$$f(y) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_n) = g(y).$$