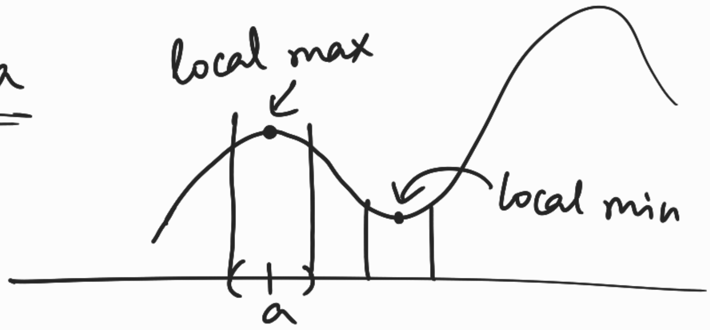
Local extrema



f is said to have a local max. at $x = a$ if $\exists\, \varepsilon > 0$ s.t. for every $x \in (a-\varepsilon, a+\varepsilon)$,

$$f(x) \leq f(a) \text{ holds.}$$

f is said to have a local min. at $x = a$ if $\exists\, \varepsilon > 0$ s.t. for every $x \in (a-\varepsilon, a+\varepsilon)$,

$$f(x) \geq f(a) \text{ holds.}$$

Fermat's theorem

Suppose that $f : [a, b] \to \mathbb{R}$ attains a local max/min at $x = c$, where $c \in (a, b)$, and $f'(c)$ exists. Then, $f'(c) = 0$.

A function which is diff'ble at exactly one point:

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Diff'ble only at $x = 0$.

if $a \neq 0$, then f not cont. at $x = a$

$$\frac{f(h) - f(0)}{h} = \begin{cases} h & \text{if } h \in \mathbb{Q} \\ 0 & \text{if } h \notin \mathbb{Q} \end{cases} \to 0$$

as $h \to 0$.

Proof of Fermat's thm:

Assume w.l.o.g. that f attains local max at $x=c$.
(Otherwise, work with $-f$.)

f attains local max. at $x=c$ means $\exists\, \delta > 0$ s.t.
for every $x \in (c-\delta, c+\delta)$, $f(x) \le f(c)$ holds.

Then for $x \in (c, c+\delta)$, $\dfrac{f(x)-f(c)}{x-c} \le 0$,

so

$$f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0. \qquad (1)$$

But on the other hand,

$x \in (c-\delta, c)$ gives $\dfrac{f(x)-f(c)}{x-c} \ge 0$

so

$$f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0. \qquad (2)$$

From (1) and (2), we conclude that

$$f'(c) = 0.$$

□

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f attains local max/min at $x=c$.

If (i) $c \in (a, b)$
(ii) $f'(c)$ exists.
then, $f'(c) = 0$.

① $f : [a, b] \to \mathbb{R}$ attains a local extremum at $c \in [a, b]$ and $f'(c)$ exists. $\overset{?}{\Longrightarrow} f'(c) = 0$.

<u>No.</u> For instance, $f(x) = x$, $x \in [0, 2]$.

f has local min at $x = 0$, and $f'(0)$ exists (one-sided)

Yet, $f'(0) = 1 \neq 0$.

② $f : [a, b] \to \mathbb{R}$, attains local extremum at $c \in (a, b)$. $\overset{?}{\Longrightarrow} f'(c)$ exists & $f'(c) = 0$

<u>No.</u> For instance, $f(x) = |x|$, $x \in [-1, 1]$

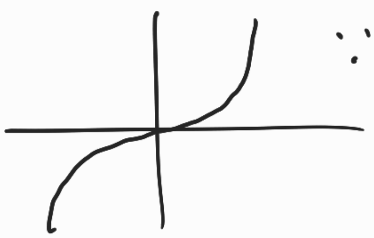
has local min at $x = 0$, but $f'(0)$ does not exist.

③ $f : [a, b] \to \mathbb{R}$, $c \in (a, b)$ s.t. $f'(c) = 0$.

$\overset{?}{\Longrightarrow} f$ has a local extremum at $x = c$.

<u>No.</u> For instance, $f(x) = x^3$, $x \in [-1, 1]$.

$f'(0) = 0$. But f neither has a local min nor a local max at $x = 0$. Why?



$\because$ for any $\varepsilon > 0$, we have $-\varepsilon/2, \varepsilon/2 \in (-\varepsilon, \varepsilon)$ s.t. $f(-\varepsilon/2) < f(0) < f(\varepsilon/2)$.

Fermat's thm

$f : [a,b] \to \mathbb{R}$

$$\left.\begin{array}{l} \text{(i) has local max/min at } x = c \\ \text{(ii) } f'(c) \text{ exists} \\ \text{(iii) } c \text{ is not an endpoint} \end{array}\right\} \begin{array}{l} \text{Then} \\ f'(c) = 0. \end{array}$$

Rolle's thm

$f : [a,b] \to \mathbb{R}$ cont. on $[a,b]$, and

diff'ble on (a,b) and $f(a) = f(b)$.

Then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

[$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

f cont. on $[0,1]$, diff'ble on $(0,1)$, but its one-sided derivative at $x = 0$ does not exist.]

Proof of Rolle's thm Since f is cont. on $[a,b]$, by extreme value thm, we can say that f must attain a max and a min inside $[a,b]$.

If any of these extrema is attained in (a,b) then we are through, by Fermat's theorem.

[Derivative of f at that point will be zero.]

If both the maximum and the minimum are attained at the endpoints, then the function must be constant ($\because f(a) = f(b)$) and the conclusion is trivial in this case. $\square$

Q. f is cont. on $[a, b]$, diff.ble on (a, b), and $f(a) = f(b) = 0$. For any $\alpha \in \mathbb{R}$, show that $\exists\, c \in (a, b)$ s.t.

$$f'(c) + \alpha f(c) = 0.$$

<u>A 'stupid' method:</u>

$$f'(x) + \alpha f(x) = 0 \text{ "\underline{for all } x\text{"}}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \int -\alpha\,dx$$

$$\Rightarrow \log f(x) = -\alpha x + c$$

$$\Rightarrow f(x) = e^{-\alpha x} \times \text{Const.}$$

$$\Rightarrow \underbrace{e^{\alpha x} f(x)}_{\text{Use this function as } g(x).} = \text{Const.}$$

<u>Rough Work</u>

<u>Sol^n</u> Apply Rolle's thm on $g(x) = e^{\alpha x} f(x)$. (Check that it works.)

Q. f is cont. on $[0, \pi]$, diff'ble on $(0, \pi)$.

Show that $\exists\ c \in (0, \pi)$ s.t.

$$f'(c) \sin c + f(c) \cos c = 0.$$

Rough work

$$\frac{f'(x)}{f(x)} = -\frac{\cos x}{\sin x} \xrightarrow{\text{integ}} \log f(x) = -\log \sin x + c$$

$$\Rightarrow \underbrace{f(x) \sin x}_{\text{This should be your } g(x).} = \text{const.}$$

Solution

Consider $g(x) = f(x) \sin x,\quad x \in [0, \pi]$.

f cont. on $[0, \pi]$, diff'ble on $(0, \pi)$, hence so is g, and $g(0) = 0 = g(\pi)$.

$\therefore$ By Rolle's thm, $\exists\ c \in (0, \pi)$ s.t.

$$g'(c) = 0,$$

i.e.,

$$f'(c) \sin c + f(c) \cos c = 0. \quad \text{(Proved)}$$

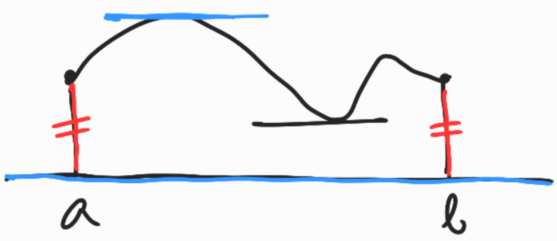
Lagrange's Mean Value Theorem (LMVT)

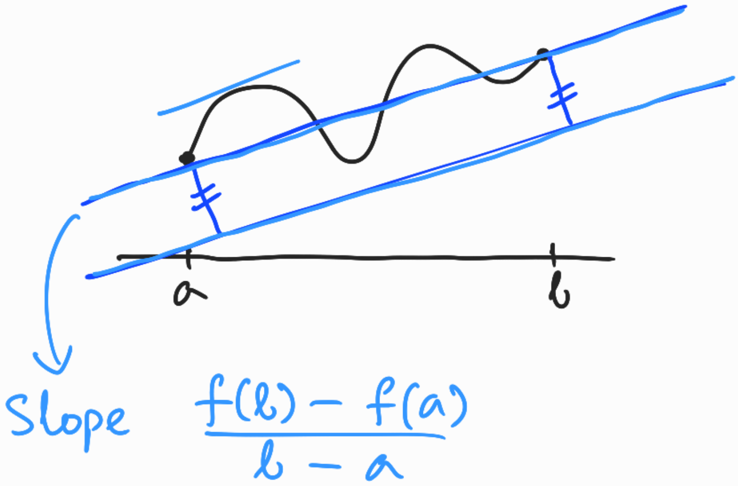
Rolle: f cont. on $[a, b]$ & $f(a) = f(b)$

diff'ble on (a, b)

What if we drop this?

Rolle's thm

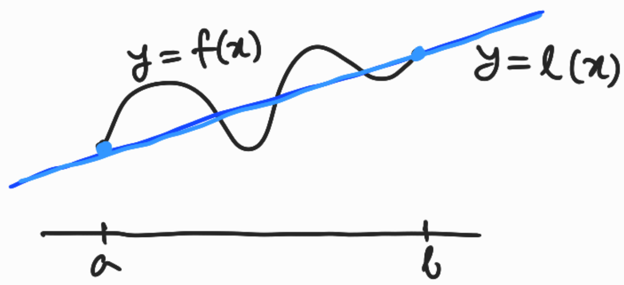




LMVT

$f:[a,b]\to\mathbb{R}$ cont. on $[a,b]$, diff.ble on (a,b). Then $\exists\, c\in(a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Proof. The line joining $(a,f(a))$ and $(b,f(b))$ is $y=l(x)$ where

$$l(x) = f(a) + (x-a)\times\frac{f(b)-f(a)}{b-a}.$$

Consider

$$g(x) = f(x) - l(x),\quad x\in[a,b].$$

g is cont. on $[a,b]$, diff.ble on (a,b), and

$$g(a) = 0 = g(b).$$

Applying Rolle's thm on g, we get $c\in(a,b)$ s.t. $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$ $\square$

Q. If $f'(x) > 0$ for all $x \in (a, b)$ and f is cont. on $[a, b]$, then f is strictly increasing. ($\underline{x < y \Rightarrow f(x) < f(y)}$)

<u>Proof.</u> Fix any $x, y \in [a, b]$, say $x < y$. Apply LMVT to f on $[x, y]$ to get

$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0$$

for some $c \in (x, y)$, and $y - x > 0$, so $f(y) > f(x)$. $\square$

Q. f diff.ble on a bounded interval I and f' bdd implies that f is also bdd.

<u>Sol^n</u> Say, $|f'(x)| \le M$ for all $x \in I$.

Assume that $I = [c, d]$ for some $c < d$.

(Other cases are similar & left as an exc.)

$$\left|\frac{f(x) - f(c)}{x - c}\right| = |f'(\xi)| \le M$$

$$\Rightarrow |f(x)| \le |f(x) - f(c)| + |f(c)|$$

$$\le M(x - c) + |f(c)| \le \underbrace{M(d - c) + |f(c)|}_{\text{Constant}}.$$