Cauchy Sequences

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June 2019

Suppose x_n is a sequence such that $\lim_{n\to\infty} x_n$ exists. Let the limit be x. Then we know that the terms are getting closer and closer to x. Now this also implies that the terms get closer and closer to each other. This intuition is made precise in the following theorem.

Theorem. If $\{x_n\}_{n\geq 1}$ is a sequence such that $\lim_{n\to\infty} x_n$ exists, then for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ holds for every $m, n \geq N$.

Proof. Let the limit be x. Fix any $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, so there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/2$ holds for every $n \geq N$. Then, for any $m, n \geq N$ we have

$$|x_m - x_n| = |(x_m - x) - (x_n - x)| \leq |x_m - x| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is exactly what we wanted to show. □

The notion for the terms of a sequence to get closer and closer to each other, as described in the above theorem, has a name:

Definition. We say that $\{x_n\}_{n\geq 1}$ is a *Cauchy sequence* if it has the following property: for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ holds for every $m, n \geq N$.

Thus, the last theorem can be restated as follows:

"Every convergent sequence must be a Cauchy sequence."

Next, we are interested to find whether the converse of the above theorem is true. That is, given a Cauchy sequence, we want to know whether it must be convergent (or not). Let us look at a simpler question first: is it necessary that every Cauchy sequence is bounded? If it seems somewhat 'trivial' to you then let me (try to) confuse you. Suppose a sequence x_n satisfies the following property: for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, $|x_{n+1} - x_n| < \varepsilon$ holds. Is it necessary that x_n is bounded?

The given property is telling us that the consecutive differences are getting smaller and smaller. So your intuition might (mis)lead you to the conclusion that the terms are getting clustered in a bounded region. Actually it turns out that such a sequence need not be bounded. Here is a counter-example: take $x_n = \sqrt{n}$ for $n \geq 1$. Then,

$$0 < x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

Therefore the sequence x_n has the property that the consecutive difference $|x_{n+1} - x_n|$ is getting smaller and smaller as n increases, but the sequence is unbounded.

Now let us come back to the question for Cauchy sequences. You must have noticed that the property that a Cauchy sequence has, is much more stronger than the property given in the last question. (Demanding $|x_m - x_n| < \varepsilon$ for every $m, n \geq N$ is much more than demanding only $|x_{n+1} - x_n| < \varepsilon$ for every $n \geq N$.) And it turns out that for x_n to be a Cauchy sequence, it must be bounded:

Theorem. If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence then it must be bounded.

Proof. Fix ε to be 1. There exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ holds for every $m, n \geq N$. So, for every $m \geq N$ we have $|x_m| \leq |x_m - x_N| + |x_N| < 1 + |x_N|$. Taking $M = \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x_N| + 1\}$, we get $|x_m| \leq M$ for all $m \geq 1$. □

You should notice that the proof for "every Cauchy sequence is bounded" is very similar to the proof for "every convergent sequence is bounded". Let us now move to the bigger question: is it necessary that every Cauchy sequence is convergent?

Before revealing the answer, let me tell you how to approach it. Suppose x_n is a Cauchy sequence. Then the last theorem says that it must be bounded. Now, being bounded does not imply it is convergent; but what can be said? We know that every bounded sequence has a convergent subsequence (Bolzano-Weierstrass theorem). Let $\{x_n\}_{n\geq 1}$ have a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to x. Then the terms of the subsequence are getting closer and closer to x. And the original sequence being Cauchy, the other terms are getting closer and closer to the terms of this subsequence. Therefore, we can see that the other terms are also getting closer and closer to x! Of course we shall make this argument precise by bringing epsilons; but it is also important for you to understand it intuitively.

Theorem. If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence then it must be convergent.

Proof. Since x_n is Cauchy, it must be bounded. Hence it has a convergent subsequence. Let $\{x_{n_k}\}_{k\geq 1}$ be that convergent subsequence, which converges to x. Now, fix any $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon/2$ holds for every $k \geq k_0$. And the sequence being Cauchy, there exists an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon/2$ holds for every $m, n \geq N$. Now, as the indices n_k are strictly increasing, there exists some $n_k > N$ with $k \geq k_0$. Then, for every $m \geq N$ we have $|x_m - x_{n_k}| < \varepsilon/2$ as well as $|x_{n_k} - x| < \varepsilon/2$. This gives us $|x_m - x| < \varepsilon$ for every $m \geq N$, which completes the proof. □

Thus, we have shown that a sequence is Cauchy if and only if convergent. Now you might ask what we gain by redefining the class of convergent sequences. The main reason is that, sometimes it is easier to show that a sequence is Cauchy than to show that it is convergent.

This is illustrated by the following theorem.

Theorem. Suppose x_n is a sequence satisfying $|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|$ for every $n > 1$, where $0 < \lambda < 1$ is fixed. Then, x_n must converge.

Proof. We shall show that x_n is a Cauchy sequence. First observe that

$$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}| \leq \lambda^2 |x_{n-1} - x_{n-2}| \leq \cdots \leq \lambda^{n-1} |x_2 - x_1|$$

which holds for all $n \geq 1$. Next, let $m > n$.

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq (\lambda^{m-2} + \lambda^{m-3} + \cdots + \lambda^{n-1})|x_2 - x_1| \\ &= \lambda^{n-1}|x_2 - x_1|(1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}) \\ &\leq \lambda^{n-1}|x_2 - x_1|(1 + \lambda + \lambda^2 + \cdots) = \lambda^n \frac{|x_2 - x_1|}{\lambda(1-\lambda)}. \end{aligned}$$

Note, in the last step, we utilised $0 < \lambda < 1$. Thus, we have shown that for every $m > n$, $|x_m - x_n| \leq c \cdot \lambda^n$ where $c = |x_2 - x_1|/\lambda(1-\lambda)$. Now, as $0 < \lambda < 1$, we know that $\lim_{n\to\infty} \lambda^n = 0$. Therefore for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $c \cdot \lambda^n < \varepsilon$ holds for every $n \geq N$. Then, it follows that for every $m, n \geq N$, $|x_m - x_n| < \varepsilon$. □

Try to prove the above theorem without using Cauchy criterion. You will understand how Cauchy criterion acts as an indispensable tool for showing convergence in some problems like the one above.

Next, we look at some other applications of Cauchy sequences.

Problem. Suppose that $a_1 = 1$ and $a_{n+1} = 1 + 1/a_n$ for all $n \geq 1$. Show that $\lim_{n\to\infty} a_n$ exists and also find this limit.

Solution. From $a_{n+1} = 1 + 1/a_n$ and $a_n = 1 + 1/a_{n-1}$, we get $|a_{n+1} - a_n| = \dfrac{|a_n - a_{n-1}|}{a_n a_{n-1}}$. Note that we used $a_n > 0$ for all $n \geq 1$ (which is quite obvious from the definition of a_n). Next, we shall give a lower bound on $a_n a_{n-1}$. (In order to arrive at something of the form $|a_{n+1} - a_n| \leq \lambda |a_n - a_{n-1}|$ where $0 < \lambda < 1$.) First observe that $a_n \geq 1$ for all $n \geq 1$. Hence, $a_n a_{n-1} = 1 + a_{n-1} \geq 2$ for every $n \geq 1$. Therefore, we get

$$|a_{n+1} - a_n| = \frac{|a_n - a_{n-1}|}{a_n a_{n-1}} \leq \frac{|a_n - a_{n-1}|}{2}, \quad \text{for all } n > 1.$$

Now, applying the previous theorem we get that $\lim_{n\to\infty} a_n$ exists. Let this limit be ℓ. Letting $n \to \infty$ in the recursion $a_{n+1} = 1 + 1/a_n$, we get $\ell = 1 + 1/\ell$. This gives $\ell = (1 \pm \sqrt{5})/2$. Since $a_n \geq 1$ for all $n \geq 1$, the limit can't be less than 1. Hence we conclude that $\lim_{n\to\infty} a_n = (1 + \sqrt{5})/2$. □

Comment. In the last problem, we evaluate first few terms of the sequence:

$$a_1 = 1, a_2 = 2, a_3 = \frac{3}{2}, a_4 = \frac{5}{3}, a_5 = \frac{8}{5}, a_6 = \frac{13}{8}, \cdots .$$

Does this ring a bell? If you know Fibonacci numbers[1], you might have guessed that $a_n = F_{n+1}/F_n$. In fact this is true and can be easily proved by induction. Therefore, what we have actually shown (in the last problem) is that,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

Can you prove this in some other way, without using Cauchy sequences?

Problem. For $n \geq 1$ define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Show that H_n is unbounded.

Solution. Observe that H_n is an increasing sequence. So in order to prove that it is unbounded, it suffices to show that it does not converge. Since every convergent sequence is Cauchy, it is enough show that H_n is not Cauchy. Can you tell what is meant by saying a sequence is not Cauchy ? If we just negate the definition of a Cauchy sequence, we get:

A sequence x_n is not Cauchy if there exists $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exists $m, n \geq N$ such that $|x_m - x_n| \geq \varepsilon_0$.

Intuitively this means that there are infinitely many terms of the sequence which are at least ε_0 apart, for some $\varepsilon_0 > 0$. In our case, we illustrate such terms of the sequence:

$$H_{2n} - H_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots (n \text{ many}) = \frac{1}{2}.$$

So, we take $\varepsilon_0 = 1/2$. For any $N \in \mathbb{N}$, we take $n = N$ and $m = 2N$ which gives $|H_m - H_n| \geq \varepsilon_0$ and we are through. □

Problem. Let x_n be any sequence. For $n \geq 1$, define $S_n = \sum_{k=1}^{n} x_k$ and $T_n = \sum_{k=1}^{n} |x_k|$. If $\lim_{n\to\infty} T_n$ exists then show that $\lim_{n\to\infty} S_n$ must exist as well.

Solution. Fix any $\varepsilon > 0$. Since T_n converges, it is also Cauchy. Hence there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, $|T_m - T_n| < \varepsilon$ holds. Now, pick any such m, n. Say $m > n$. Then, observe that

$$|S_m - S_n| = \left| \sum_{k=n+1}^{m} x_k \right| \leq \sum_{k=n+1}^{m} |x_k| = T_m - T_n < \varepsilon.$$

This proves that the sequence S_n is Cauchy and hence convergent. □

[1] n-th Fibonacci number F_n is defined as: $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.

<u>Exercises</u>

1. Suppose x_n is a sequence satisfying $|x_{n+1} - x_n| \le \dfrac{1}{2^n}$ for all $n \ge 1$. Show that $\lim\limits_{n\to\infty} x_n$ exists.

2. For $n \ge 1$, define $x_n = \displaystyle\sum_{k=1}^{n} \frac{\sin k}{k^2}$. Show that $\{x_n\}_{n\ge1}$ is a Cauchy sequence.

3. Let $\{x_n\}_{n\ge1}$ be a sequence satisfying $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for every $n \ge 1$. Is it necessary that $\{x_n\}_{n\ge1}$ converges?

4. Let $\{x_n\}_{n\ge1}$ be a Cauchy sequence. Is it necessary that there exists some $\lambda \in (0,1)$ such that $|x_{n+1} - x_n| \le \lambda |x_n - x_{n-1}|$ holds for every $n > 1$?

5. Suppose that $1 \le x_1 \le x_2 \le 2$ and define $x_{n+2} = \sqrt{x_{n+1}x_n}$ for $n \ge 1$. Show that,

   (a) $x_{n+1}/x_n \le 2$ holds for every $n \ge 1$.

   (b) $|x_{n+2} - x_{n+1}| \le \frac{2}{3}|x_{n+1} - x_n|$ holds for every $n \ge 1$.

   (c) Hence conclude that $\lim\limits_{n\to\infty} x_n$ exists.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \le \dfrac{1}{2}|x - y|$ for all $x, y \in \mathbb{R}$. Show that the equation $f(x) = x$ has a unique solution.

7. Take any $a > 0$ and define a sequence $\{x_n\}_{n\ge1}$ as follows: start with any $x_1 > 0$ and set $x_{n+1} = a/(1 + x_n)$ for every $n \ge 1$. Show that $\lim_{n\to\infty} x_n$ exists, and is a root of the equation $x^2 + x = a$.