L'Hôpital's rule

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What many think of as L'Hôpital's rule is the following:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}. \qquad (*)$$

Non-examples

① $\displaystyle\lim_{x\to 0}\frac{x}{x+1} \overset{(*)}{=} \lim_{x\to 0}\frac{1}{1+0} = 1$

But $\displaystyle\lim_{x\to 0}\frac{x}{x+1} = 0.$ ☹

Not $0/0$ or $\pm\infty/\infty$ form. Hence $(*)$ does not apply.

② $f(x) = \begin{cases} x^2\sin(x^{-2}), & x\neq 0 \\ 0, & x = 0. \end{cases}$ $\quad g(x) = x.$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} \overset{(*)}{=} \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{\boxed{2x\sin(x^{-2})} + \boxed{2x^{-1}\cos(x^{-2})}}{1}$$

limit $= 0$. (by Sandwich) $\qquad$ does not have a limit.

But,

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0} x\sin(x^{-2}) = 0 \text{ (by Sandwich)}$$

Then when does (*) hold??

① limit must be of the form f/g where

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \pm\infty.$$

② f', g' must exist in a nbd of $x=a$, except possibly at $x=a$.

③ $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, say equals ℓ.

$$\boxed{\text{If ①, ② \& ③ hold, can we say that } \lim_{x\to a}\frac{f(x)}{g(x)} \text{ exists and equals } \ell\text{?}}$$

Another non-example:

$$f(x) = x + \sin x \cos x, \quad g(x) = f(x)\, e^{\sin x}.$$

$$\lim_{x\to\infty} f(x) = \infty. \quad \lim_{x\to\infty} g(x) = \infty.$$

$$\underbrace{(x}_{\to\infty} + \underbrace{\sin x \cos x}_{bdd})\, \underbrace{e^{\sin x}}_{>0}$$

① $\frac{\infty}{\infty}$ form

② f', g' exist

$$f'(x) = 1 + \cos 2x,$$

$$g'(x) = (1 + \cos 2x + \cos x\,(x + \sin x \cos x))\, e^{\sin x}.$$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{(1+\cos 2x)\, e^{-\sin x}}{1+\cos 2x + \cos x\,(\boxed{x} + \sin x \cos x)}$$

$$= 0 \qquad \frac{1}{\infty} = \frac{1}{-\infty} = 0$$

③ $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists and equals 0.

①, ② & ③ hold for this example. But,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-\sin x} \rightarrow \text{does not exist.}$$

We need one more assumption for L'Hôpital's rule.

④ $g'(x) \neq 0$ in a nbd of $x=a$, except possibly at $x=a$.

Here,

$$g'(x) = (1+\cos 2x + \cos x\,(x+\sin x \cos x))\, e^{\sin x}.$$

$$g'(x) = 0 \quad \text{whenever} \quad x = 2n\pi + \frac{\pi}{2},\ n \in \mathbb{N}.$$

$\therefore$ For no $M>0$ we have $g' \neq 0$ on (M, ∞).

That's why one cannot apply L'Hôpital's rule here.

- It turns out that ① through ④ are sufficient.
- When $\lim\limits_{x\to a} g(x) = \pm\infty$, we need no assumption on limit of $f(x)$.

$$\begin{array}{ccc} \text{Johann Bernoulli} & \xleftarrow{\text{money}} & \text{L'Hôpital} \\ \downarrow & \xrightarrow[\text{\& fame}]{\text{credit}} & \text{(See: Wikipedia)} \\ \text{who invented} & & \end{array}$$

L'Hôpital's rule (version 1)

Let $-\infty \le a < b \le \infty$ and f, g be diff'ble on (a, b). Suppose that,

① $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$, ("$0/0$ form")

② $g' \neq 0$ on (a, b),

③ $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists and equals ℓ, where $\ell \in \mathbb{R}$ or $\ell = \pm\infty$.

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and also equals ℓ.

L'Hôpital's rule (version 2)

Just replace condition ① in version 1 with

①' $\lim_{x \to a^+} g(x) = \pm\infty$. Then also we get the same conclusion.

[The same is true if you replace "$x \to a^+$" with $x \to b^-$ or $x \to c$ where $c \in (a, b)$.]

Proof of version 1

Case 1: $\ell \in \mathbb{R}$ " $\lim_{x \to a^+} h(x) = \ell$ " **def^n**:

for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.
$\ell - \varepsilon < h(x) < \ell + \varepsilon$ holds whenever $a < x < a + \delta$.

This does not work when $a = -\infty$.

" $\lim_{x \to -\infty} h(x) = \ell$ " def^n:

for every $\varepsilon > 0$ there exists $m \in \mathbb{R}$ s.t.
$\ell - \varepsilon < h(x) < \ell + \varepsilon$ holds whenever $x < m$.

$\boxed{\text{Can we give a unified definition?}}$

$\lim_{x \to a^+} h(x) = \ell$ def^n (where $-\infty \leq a < \infty$):

for every $\varepsilon > 0$ there exists $m > a$ s.t.
$\ell - \varepsilon \leq h(x) \leq \ell + \varepsilon$ for all $x \in (a, m)$.

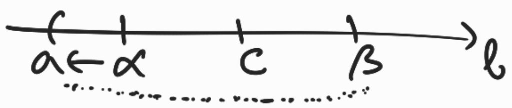
Fix $a < \alpha < \beta < b$. By Cauchy's MVT,

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)}$$

Here we use the assump. that $g' \neq 0$.

for some $c \in (\alpha, \beta)$.

Idea: $\alpha \to a^+$. Issue: No control over c.



Since $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}$, for every $\varepsilon > 0$, there exists $m > a$ s.t. for every $x \in (a, m)$ we have

$$\ell - \varepsilon \leq \frac{f'(x)}{g'(x)} \leq \ell + \varepsilon.$$

Fix α, β s.t. $a < \alpha < \underline{\beta < m} < b$.

Then for some $\alpha < \underline{c < \beta < m}$

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)} \in [\ell - \varepsilon, \ell + \varepsilon].$$

Thus, for any $a < \alpha < \beta < m$,

$$\ell - \varepsilon \leq \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} \leq \ell + \varepsilon.$$

Let $\alpha \to a^+$ and use $\lim\limits_{\alpha\to a^+} f(\alpha) = 0 = \lim\limits_{\alpha \to a^+} g(\alpha)$ to deduce that

$$\ell - \varepsilon \leq \frac{f(\beta)}{g(\beta)} \leq \ell + \varepsilon$$

for every $\beta \in (a, m)$.

$\Downarrow$ (from the unified def^n)

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \ell.$$

Case 2: $l \in \{\infty, -\infty\}$. Say, $l = \infty$ (w.l.o.g.)

$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \infty$ def^n (unified):

$\forall M > 0 \ \exists u > a$ s.t. for every $x \in (a, u)$,

$$\frac{f'(x)}{g'(x)} \geq M.$$

Fix α, β s.t. $a < \alpha < \beta < u < b$. Applying CMVT, we get

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)} \geq M.$$

Now let $\alpha \to a^+$ to conclude that for every $\beta \in (a, u)$,

$$\frac{f(\beta)}{g(\beta)} \geq M.$$

By the unified def^n, this means that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty.$$

This completes the proof of version 1.

Proof of v2 is similar.

(See Bartle & Sherbert)

Examples

① $$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}.$$

✓ $0/0$ form

✓ f', g' exist

✓ $g' \neq 0$ on, say, $(-\pi/2, \pi/2)$

$f(x) = 2\sin x - \sin 2x$,

$g(x) = x - \sin x$,

$g'(x) = 1 - \cos x$.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \cdots = l$$

Since the latter limit exists and equals l, so does the former, by L'Hôpital's rule.

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x} = \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x}$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x}$$

We can again apply L'Hôpital's rule

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{\cos x}$$

$$= \frac{-2+8}{1} = 6.$$

$\therefore$ By L'Hôpital's rule, the desired limit is 6. (Ans)

② $\lim_{x\to 0^+} x^x = ?$

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x}$$

✓ $\frac{-\infty}{\infty}$ form

✓ num. & denom. diff'ble

✓ $\left(\frac{1}{x}\right)' = \frac{-1}{x^2} \neq 0$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} (-x) = 0.$$

$\therefore$ By L'Hôpital's rule, $\lim_{x\to 0^+} x\log x = 0.$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^{\lim_{x\to 0^+} x\log x}$$

$$= e^0 = 1. \text{ (Ans)}$$

($0^0 \overset{\text{def}}{=} 1$ in many places.)

③ For any $n\in\mathbb{N}$, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n x^{n-1}}{e^x}$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$

$$\vdots$$

$$= \lim_{x\to\infty} \frac{n!}{e^x}$$

(By L'Hôpital's rule)

Constant

In each step,

- ∞/∞ form,
- num & denom diff'ble,
- $(e^x)' = e^x \neq 0$

$= 0.$ Hence proved.

$$\boxed{\lim_{x\to\infty} \frac{\text{Poly}(x)}{e^x} = 0.}$$

Don't apply L'Hôpital's rule blindly, even if all the requirements are satisfied:

① $$\lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \overset{L}{=} \lim_{x\to\infty} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \cdots.$$

② $$\lim_{x\to\infty} \frac{x^{1/2}-x^{-1/2}}{x^{1/2}+x^{-1/2}} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-3/2}+\frac{1}{2}x^{-5/2}}{\frac{1}{2}x^{-3/2}-\frac{1}{2}x^{-5/2}}$$

$\overset{L}{=} \cdots$ It gets worse.

$$\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = f''(x).$$

Suppose $f''(x)$ exists. Then

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

exists and equals $f''(x)$. Proof? Use L'Hôpital's rule.

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

$$= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \quad \text{(by L'Hôpital's rule)}$$

Add and subtract $f'(x)$ and complete. $\square$