Problems on L'Hôpital's rule

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1. Evaluate the following limits

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right), \quad \lim_{x\to\infty}(e^x+x)^{1/x}, \quad \lim_{x\to 0+}(\cos 2x)^{1/x^2}.$$

2. Determine, with proof, the value of $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}$ and $\lim_{x\to 0}\frac{x-\sin x}{x^3}$.

3. Evaluate $\lim_{x\to 0+} x^x$ and hence justify what should be the convention for the value of 0^0 (in order to make the function x^x right-continuous at 0).

4. Let f, g be functions defined on $[a, b]$ and $f(a) = g(a) = 0$. Suppose that f, g are both differentiable at a. Furthermore, assume that $g(x) \neq 0$ for $a < x < b$ and $g'(a) \neq 0$. Show that $\lim_{x\to a} f(x)/g(x)$ exists and equals $f'(a)/g'(a)$.

5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) = 0$, and let $g(x) = x^2$ for $x \in \mathbb{R}$. Show that $\lim_{x\to 0} f(x)/g(x)$ does not exist. Find out why the result of the last problem does not hold here.

6. Let $f(x) = x^2$ for x rational, let $f(x) = 0$ for x irrational, and let $g(x) = \sin x$ for $x \in \mathbb{R}$. Find out why we can't use the standard L'hôpital's rule for this problem. Use one of the above problems to evaluate $\lim_{x\to 0} f(x)/g(x)$.

7. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) = 0$, and let $g(x) = \sin x$ for $x \in \mathbb{R}$. Show that $\lim_{x\to 0} f(x)/g(x) = 0$ but that $\lim_{x\to 0} f'(x)/g'(x)$ does not exist.

8. For each of the following examples, explain why L'hôpital's rule can't be applied to write $\lim_{x\to\infty} f(x)/g(x) = \lim_{x\to\infty} f'(x)/g'(x)$.

   (a) $f(x) = 4 + 1/x$, and $g(x) = 2 + 1/x$.

   (b) $f(x) = x + \sin x$, and $g(x) = x$.

   (c) $f(x) = x + \sin x \cos x$, and $g(x) = f(x)e^{\sin x}$.

9. Let f be a differentiable function such that

$$\lim_{x\to\infty}(f(x) + f'(x)) = \ell \in \mathbb{R}.$$

   Show that $\lim_{x\to\infty} f'(x) = 0$. If we drop the assumption that ℓ is finite, will the conclusion still hold?

10. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that, $\lim_{x\to\infty} f(x) = 2019$. Show that if $\lim_{x\to\infty} f'(x)$ exists then it must be zero. Is it necessary that $\lim_{x\to\infty} f'(x)$ exists?

11. Determine, with proof, the value of

$$\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right).$$

12. Suppose that f is twice differentiable in a neighbourhood of c. Show that

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = f''(c).$$

13. Suppose that f is a function such that the limit

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$$

exists and is finite. Does this imply that $f''(c)$ exists and equals the above limit?

14. Show that for any polynomial $P(x)$ with real coefficients, $\lim_{x\to\infty} P(x)/e^x = 0$.

15. Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$. Determine, with proof, the value of $f^{(n)}(0)$ for every $n \ge 1$. (Here $f^{(n)}(0)$ denotes the n-th derivative of f, evaluated at 0.)

16. Suppose that f is a function such that $f''(a)$ exists. Define a quadratic polynomial $p(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2$. Show that

$$\lim_{x\to a} \frac{f(x)-p(x)}{(x-a)^2} = 0.$$

(Note that in order to have f twice differentiable at $x = a$, we need $f'(x)$ to be defined in a open neighbourhood of a, i.e. f must be differentiable in a open neighbourhood of a.)

For more practice problems, visit: `https://www.math24.net/lhopitals-rule/` and `http://math.cmu.edu/~bkell/lhopital.pdf`.