L'Hôpital's rule – Problems

#8(b) $f(x) = x + \sin x, \quad g(x) = x$

Why L'Hôpital's rule can't be applied to write

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \;?$$

- f, g diff'ble on $\mathbb{R}$
- $g' \neq 0$ on $\mathbb{R}$
- ∞/∞ form.

But $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \left(\frac{1+\cos x}{1}\right)$ ← does not exist.

That's why we cannot write

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

#9. f diff'ble, $\displaystyle\lim_{x\to\infty} \left(f(x) + f'(x)\right) = \ell \in \mathbb{R}$.

Show that $\displaystyle\lim_{x\to\infty} f'(x) = 0$.

(i.e., show that this limit exists & equals 0.)

$$\frac{d}{dx}\left(e^x f(x)\right) = e^x\left(\underline{f(x) + f'(x)}\right).$$

$$f(x) + f'(x) = \frac{\frac{d}{dx}\left(e^x f(x)\right)}{\frac{d}{dx} e^x}$$

$g(x) = e^x f(x), \quad h(x) = e^x.$

$$\boxed{\lim_{x\to\infty} \frac{g(x)}{h(x)} \longleftarrow \frac{\cdot}{\infty} \text{ form } \quad \text{L'Hôpital's v2 needed}}$$

✓ $\lim_{x\to\infty} h(x) = \infty$

✓ g, h diff'ble, and $h'(x) \neq 0$ for any x.

✓ $\lim_{x\to\infty} \frac{g'(x)}{h'(x)} = \lim_{x\to\infty} \frac{\cancel{e^x}(f(x)+f'(x))}{\cancel{e^x}}$

$= \ell \in \mathbb{R}$ (given).

$\therefore$ By L'Hôpital's rule, we can say that

$\lim_{x\to\infty} \frac{g(x)}{h(x)}$ must exist and also equal ℓ.

But note that $\frac{g(x)}{h(x)} = f(x)$. So, we got

the conclusion that

$$\lim_{x\to\infty} f(x) = \ell.$$

If $\ell = \pm\infty$, upto this will hold true.

$$\therefore \lim_{x\to\infty} f'(x) = \lim_{x\to\infty} (f(x) + f'(x) - f(x))$$

$$= \ell - \ell = 0. \quad \text{(Proved)}$$

If l is not finite, then the above conclusion does not hold. Take, e.g.,

$$f(x) = x^2 \text{ or } e^x.$$

Try this similar problem:

$$\lim_{x\to\infty} f(x) = l \in \mathbb{R}, \quad \lim_{x\to\infty} f^{(n)}(x) = 0$$

$$\Rightarrow \lim_{x\to\infty} f^{(k)}(x) = 0, \quad 1 \le k \le n-1.$$

e.g. $n=2$ $\quad \lim_{x\to\infty} f(x) = l, \ \lim_{x\to\infty} f''(x) = 0.$

$$g(x) = f(x) - f'(x).$$

$$\lim_{x\to\infty} \left(g(x) + g'(x)\right) = \lim_{x\to\infty} \left(f(x) - f''(x)\right)$$

$$= l.$$

$\therefore$ Done by prev. problem.

In general, we should try induction on $n > 1$.

But finding the general form is difficult.

#10. $\lim_{x\to\infty} f'(x) = \ell$ exists (can be infinite also).

By the method of the last problem (#9), we can say that

$$\underbrace{\lim_{x\to\infty} f(x)}_{2019} = \underbrace{\lim_{x\to\infty} \left(f(x) + f'(x)\right)}_{2019+\ell}$$

$$\Rightarrow \ell = 0.$$

<u>Counter-example for second part:</u>

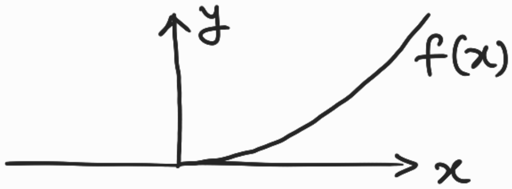
$\lim_{x\to\infty} f(x) = 2019$. $\lim_{x\to\infty} f'(x)$ does not exist.

$$f(x) = 2019 + \frac{\sin(x^2)}{x}$$

Clearly, (by Sandwich) $\lim_{x\to\infty} f(x) = 2019$.

$$f'(x) = \underbrace{\frac{-1}{x^2}\sin(x^2)}_{\to 0 \text{ as } x\to\infty} + \underbrace{2\cos(x^2)}_{\text{does not have a limit as } x\to\infty}$$

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$



#15. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$\lim\limits_{x \to 0} f(x) = 0$. (See why.) $\therefore$ f is cont.

What about $f'(0)$?

$\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = 0$. What about $x \to 0^+$?

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{e^{-1/x}}{x}$$

$$\left[\frac{1}{x} = y\right] \qquad = \lim_{y \to \infty} \frac{y}{e^y} = 0.$$

$$\therefore \quad f'(x) = \begin{cases} \frac{1}{x^2} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

What about $f''(0)$? $f^{(3)}(0)$?

$$f''(x) = \begin{cases} e^{-1/x}\left(\frac{-2}{x^3} + \frac{1}{x^2}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

<u>Claim</u> For any $n \geq 0$,

$$f^{(n)}(x) = \begin{cases} e^{-1/x} P_n\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

for some poly. $P_n(\cdot)$.

We prove this claim by induction on n.
The only hurdle is at $x = 0$.
Suppose the claim holds for all $n \leq m$.

$$\lim_{x \to 0^+} \frac{f^{(m)}(x) - f^{(m)}(0)}{x - 0}$$ (left hand limit is clearly zero)

$$= \lim_{x \to 0^+} \frac{e^{-1/x} P_m\left(\frac{1}{x}\right)}{x}$$

$$= \lim_{y \to \infty} \frac{y P_m(y)}{e^y} \qquad \left(y = \frac{1}{x}\right)$$

$= 0$ (by a problem discussed earlier).

Thus our induction is complete. (check)

<u>Conclusion</u> $f^{(n)}(0) = 0$ for all $n \geq 0$.

<u>Remark.</u> Even though f (as defined in the last problem) is infinitely diff'ble, but

$$f(x) \neq \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$

for any $x > 0$.

<u>#16.</u> f is a fn. such that $f''(a)$ exists.

$$P(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2}$$

Show that

$$\lim_{x \to a} \frac{f(x) - P(x)}{(x-a)^2} = 0.$$

$$P(a) = f(a),\ P'(a) = f'(a),\ P''(a) = f''(a).$$

$$\lim_{x \to a} \frac{f(x) - P(x)}{(x-a)^2}$$

✓ $0/0$ form

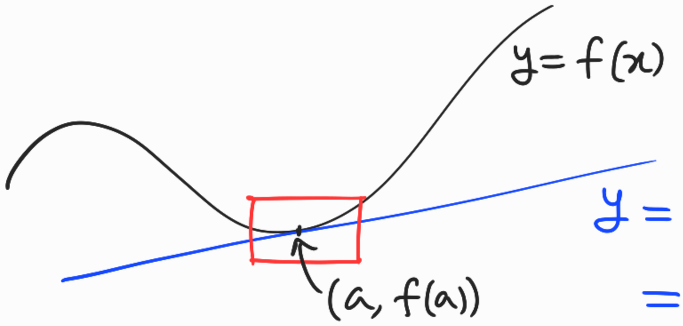
✓ num. & denom. diff'ble

✓ $(\text{denom})' \neq 0$ except at a

$$= \lim_{x \to a} \frac{f'(x) - P'(x)}{2(x-a)}$$

$$= \lim_{x \to a} \frac{(f'(x) - f'(a)) - (P'(x) - P'(a))}{2(x-a)}$$

$$= \frac{1}{2}(f''(a) - P''(a)) = 0.$$



$$\frac{y-f(a)}{x-a} = f'(a)$$

$$y = f(a) + f'(a)(x-a)$$
$$= P_1(x)$$

$$\underline{\underline{f'(a) \text{ exists}}} \Rightarrow \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)-P_1(x)}{x-a} = 0.$$

This tells us that the error in the linear approx.

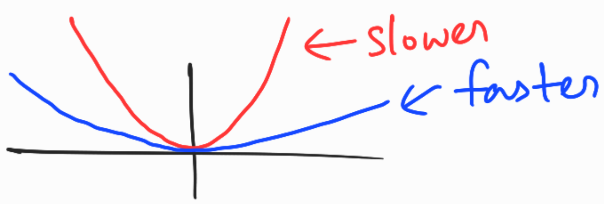
$$f(x) \approx f(a) + f'(a)(x-a)$$

not only goes to zero as $x \to a$, but also goes to zero at a rate faster than that of $(x-a)$.

$(x-a)^2 \to 0$ "faster than" $(x-a) \to 0$.

$\sqrt{|x-a|} \to 0$ "slower than" $(x-a) \to 0$.

$(x-a)\sin(x-a)$ "faster than" $(x-a) \to 0$.



"As $x \to a$, $f(x) \to 0$ faster than $g(x) \to 0$"

if $f(x)/g(x) \to 0$ as $x \to a$.

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!}$$

$f'(a)$ exists $\Rightarrow$ $f(x) = P_1(x) + e_1(x)$

where $\dfrac{e_1(x)}{x-a} \to 0$ as $x \to a$.

$f''(a)$ exists $\Rightarrow$ $f(x) = P_2(x) + e_2(x)$

where $\dfrac{e_2(x)}{(x-a)^2} \to 0$ as $x \to a$.

$f^{(n)}(a)$ exists $\Rightarrow$ $\boxed{f(x) = P_n(x) + e_n(x)}$

where $\dfrac{e_n(x)}{(x-a)^n} \to 0$ as $x \to a$.

<u>Taylor</u> If $f^{(n+1)}$ also exists, then

$$e_n(x) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}$$

for some c.

Taylor's theorem

Assume that f is $(n+1)$ times diff'ble on (a,b) and $c \in (a,b)$. Then, for any $x \in (a,b)$,

$$f(x) = \sum_{k=0}^{n} f^{(k)}(c) \frac{(x-c)^k}{k!} + e(x)$$

where

$$e(x) = f^{(k+1)}(\xi) \frac{(x-c)^{n+1}}{(n+1)!}$$

for some ξ between c and x.

$f(x) = e^x$. $c = 0$. Fix a point y.

By Taylor's thm,

$$f(y) = \sum_{k=0}^{n} \underbrace{f^{(k)}(0)}_{(=1)} \frac{y^k}{k!} + f^{(n+1)}(\xi_n) \frac{y^{n+1}}{(n+1)!}$$

$$\Rightarrow \left| e^y - \sum_{k=0}^{n} \frac{y^k}{k!} \right| = \left| e^{\xi_n} \frac{y^{n+1}}{(n+1)!} \right|$$

$|\xi_n| \le |y|$

let $R > 0$ be s.t. $|y| \le R$.

$$\le e^R \frac{R^{n+1}}{(n+1)!}$$

(Why?) $\to 0$, as $n \to \infty$.

[For fixed $R > 0$, $\lim_{n\to\infty} \frac{R^n}{n!} = 0$. Why?

$\sum_{n=0}^{\infty} \frac{R^n}{n!}$ Convergent (e.g., by ratio test)

Hence $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.]

$\sum a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

Also,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \lim_{n\to\infty} |a_n| = 0$.

Conclusion For any fixed $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \left| e^y - \sum_{k=0}^{n} \frac{y^k}{k!} \right| = 0.$$

Hence,

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}, \quad y \in \mathbb{R}.$$

Using the same recipe, find the Taylor series for

$\sin x$, $\cos x$, $\frac{1}{1-x}$, $\frac{1}{1+x}$,

$\log(1 \pm x)$, $\sqrt{1-x}$ etc.

(for appropriate values of x)