Convex Functions

Aditya Ghosh

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Warm-up problem: Suppose that f is twice differentiable in a neighbourhood of c. Show that the limit

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$$

exists and equals $f''(c)$. On the other hand, does the existence of the above limit imply that $f''(c)$ exists and equals the above limit?

Definition 1 (Convex sets)**.** A set $S \subset \mathbb{R}^d$ $(d \geq 1)$ is called convex if for every $x, y \in S$ and any $t \in [0,1]$, $tx + (1-t)y \in S$. In other words, for every two points in the set, the line segment connecting those points lies completely inside S.

Examples. Any interval in $\mathbb{R}$, any straight line in $\mathbb{R}^2$, any disc in $\mathbb{R}^2$, etc.

Definition 2 (Convex functions)**.** Let X be a convex subset of $\mathbb{R}^d$ and let $f : X \to \mathbb{R}$ be a function. Then f is called convex if, for all $0 \leq \lambda \leq 1$ and $x_1, x_2 \in X$, it holds that

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

f is called strictly convex if the above inequality is strict for every $x \neq y$ and any $\lambda \in [0,1]$.

Definition 3 (Concave functions)**.** Let X be a convex subset of $\mathbb{R}^d$ and let $f : X \to \mathbb{R}$ be a function. Then f is called concave if, for all $0 \leq \lambda \leq 1$ and $x_1, x_2 \in X$, it holds that

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \lambda f(x_1) + (1-\lambda) f(x_2).$$

f is called strictly concave if the above inequality is strict for every $x \neq y$ and any $\lambda \in [0,1]$.

Obviously, f is concave if and only if $-f$ is convex.

In case you are intimidated by the presence of $\mathbb{R}^d$, don't worry, it was mentioned only for telling the intuition of *line segment joining x and y*. Hereafter we shall focus on convex/concave functions of one variable only.

Convince yourself that convex subsets of $\mathbb{R}$ are only the intervals (open/closed/semi-open/semi-closed/bounded/unbounded) or singleton sets (which can also be thought as degenerate intervals: $[a,a] = \{a\}$).

It is not hard to show that $|x|, ax+b$ are convex functions. But proving that a given function is convex by verifying the above definition is undoubtedly difficult, in general. So we derive the following tools that would help us for proving convexity.

Problem 1. Show that a twice differentiable function defined on an open interval $I \subset \mathbb{R}$ is convex if and only if its second derivative is non-negative on I.

The test given by Problem 1 is frequently used for checking convexity. Note that if the second derivative is positive at all points then the function is strictly convex, but the converse does not hold — can you give any such example?

Problem 2. Show that a differentiable function defined on an open interval $I \subset \mathbb{R}$ is convex if and only if its derivative is non-decreasing on I.

Examples. Convince yourself that the following are true.

- x^2, x^4 are convex on $\mathbb{R}$.
- x^3, x^5 are convex on $[0, \infty)$ and concave on $(-\infty, 0]$. The point $x = 0$ where these functions make the transition from concavity to convexity (or vice-versa) is called an *inflection point.*
- $\sin x$ is concave on $[0, \pi]$, $\cos x$ is concave on $[0, \pi/2]$.
- $\log x, \sqrt{x}$ are concave on $(0, \infty)$.
- $f(x) = 1/x$ is convex on the interval $(0, \infty)$ and concave on the interval $(-\infty, 0)$.
- Define $f(x) = 0$ if $x \in (0, 1)$ and 1 if $x = 0, 1$. Then f is convex on $[0, 1]$.

Problem 3. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Show that f is convex if and only if for all $x, y, z \in I$, say $x < y < z$, it holds that

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Problem 4. Show that a convex function f defined on some open interval $I \subset \mathbb{R}$ must be continuous on I. If I is closed, then f may fail to be continuous at the endpoints of I — can you give any such example?

Problem 5. If $f : \mathbb{R} \to \mathbb{R}$ is a convex function and $f(0) \leq 0$, show that $f(a + b) \geq f(a) + f(b)$ holds for every $a, b > 0$.

Problem 6. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is a convex function, with $\lim_{x \to 0} f(x) = 0$. Prove that $g(x) = f(x)/x$ (defined for $x > 0$) is increasing.

Problem 7 (Jensen's inequality)**.** Let $f : X \to \mathbb{R}$ be convex. Then for any $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$, prove that

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

Problem 8. As an application of Jensen's inequality, prove the (simple) AM–GM inequality, the weighted AM–GM inequality, or more generally, the power mean inequality.

Problem 9. If A, B, C be the angles of a triangle, show that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

Problem 10. If A, B, C be the angles of an acute triangle, show that

$$\cos A \cos B \cos C \leq \frac{1}{8}.$$

Problem 11. Let $f : [a, b] \to \mathbb{R}$ be convex. Then show that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that

$$\frac{1}{2y}\int_{x-y}^{x+y} f(t)\,dt = f(x)$$

for all $x \in \mathbb{R}$ and $y > 0$. Show that there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

Problem 13. Let $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1 + a_2 + \cdots + a_n = 1$. Prove that for any $k \in \mathbb{N}$,

$$\left(a_1^k + \frac{1}{a_1^k}\right)\left(a_2^k + \frac{1}{a_2^k}\right)\ldots\left(a_n^k + \frac{1}{a_n^k}\right) \geq \left(n^k + \frac{1}{n^k}\right)^n.$$

Problem 14. Let $f : (a, b) \to \mathbb{R}$ be a continuous function that satisfies

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for every $x, y \in (a, b)$. Show that f must be a convex function.

Hints: First show that

$$f\Big(\frac{x_1 + \cdots + x_k}{k}\Big) \leq \frac{f(x_1) + \cdots + f(x_k)}{k}$$

for all $x_i \in (a, b)$, and for every $k = 2^m$, $m \geq 1$. Next prove by induction that the above for any k (further hint: consider $2^{m-1} \leq k < 2^m$). Next, show that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for any $x, y \in (a, b)$ and λ of the of the form $\{k/2^n : k = 0, 1, \ldots, 2^n, n \geq 1\}$. Numbers in this set are called *dyadic rationals* in $[0, 1]$. Now show that for any real number $\lambda \in [0, 1]$ we can find a sequence of dyadic rationals that converge to λ. Hence complete the proof.